Spectrum of the Hermitian Wilson-Dirac operator for a uniform magnetic field in two dimensions

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The spectrum of the Hermitian Wilson-Dirac operator is investigated for an arbitrary uniform magnetic field in two dimensions. It can be described by a relativistic analogue of the Harper equation. The index of the overlap Dirac operator is obtained directly from the spectral asymmetry of the relativistic Harper system. It coincides with the topological charge if the field strength is equal to or less than $\pi/2$.

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Topologically nontrivial gauge field configurations are responsible for various nonperturbative phenomena in QCD and the standard model. The key relation is the chiral anomaly and the index theorem. Lattice regularization is the most promising approach for nonperturbative studies and, hence, the topological structure of the continuum gauge fields should be retained also in lattice theories. This can be achieved by imposing a kind of smoothness condition on the link variables [1]. It is indeed possible to define a topological charge geometrically for sufficiently smooth link variables [1,2]. The index of the lattice Dirac operator would coincide with the topological charge as the index theorem dictates. On the lattice, however, it is difficult to compute the index explicitly even for the Abelian gauge background [3] and establishing the index theorem on finite lattices is rather nontrivial.

In this article we investigate the spectrum of the Hermitian Wilson-Dirac operator (HWDO) for a uniform magnetic field in two dimensions and determine the index of the overlap Dirac operator (ODO) [4] by directly computing the spectral asymmetry. The spectral flow of the HWDO for a one-parameter family of link variables was investigated in Ref. [5] and the connection to the chiral anomaly was elucidated. Recently, one of the present authors reanalyzed a similar system [6] and gave some exact results on the spectrum for a particular set of uniform magnetic fields, for which the index of the ODO can be obtained rigorously. We extend this to an arbitrary uniform magnetic field and show that the index coincides with the topological charge if the magnetic field is equal to or less than[1] $\pi/2$. The key idea is that the two-dimensional system described by the HWDO for a uniform magnetic field can be converted to one-dimensional lattice systems[2] described by a relativistic analogue of the Harper equation [10] and the spectrum can be characterized by a set of polynomials defined by the secular determinant of the relativistic Harper system. When the topological charge and the lattice size have a common divisor, they satisfy a factorization property, which not only plays a crucial role in determining the index but also is useful in understanding the fractal structure of the spectrum. The emergence of the fractal structure is not surprising since HWDO is a relativistic analogue of the Hamiltonian of tightly bounded electron, for which the fractal structure of the energy band is as well known as the butterfly diagram [11].

We consider a two-dimensional square lattice of unit lattice spacing and of sides L. The HWDO H is defined by

$$H\psi(x)=\sigma_3\left\{(2-m)\psi(x)-\sum_{\mu=1,2}\left(\frac{1-\sigma_\mu}{2}U_\mu(x)\psi(x+\hat{\mu})+\frac{1+\sigma_\mu}{2}U_\mu^*(x-\hat{\mu})\psi(x-\hat{\mu})\right)\right\}, \tag{1}$$

where σ_μ $(\mu=1,2,3)$ are the Pauli matrices and $\hat{\mu}$ stands for the unit vector along the μth axis. The link variables $U_\mu(x)$ and the lattice fermion field $\psi(x)$ are subject to periodic boundary conditions. In the context of ODO m must be chosen to satisfy $0<m<2$ in order for the correct continuum limit to be achieved. We adopt $m=1$ unless otherwise specified.

We are interested in the spectrum of H for the link variables of the form

$$U_1(x)=\exp\left[-it\frac{2\pi}{L}\overline{x_2}\delta_{\overline{x_1},L-1}\right],\quad U_2(x)=\exp\left[it\frac{2\pi}{L^2}\overline{x_1}\right], \tag{2}$$

where t is an arbitrary parameter and $\overline{x_\mu}$ stands for the periodic lattice coordinates defined by $\overline{x_\mu}=x_\mu$ for $0\leqslant x_\mu<L$ and $\overline{x_\mu+L}=\overline{x_\mu}$. For $t=Q$ being an integer, the magnetic field $F_{12}(x)=-i\ln U_1(x)U_2(x+\hat{1})U_1^*(x+\hat{2})U_2^*(x)$ becomes a constant and the flux per plaquette in units of 2π is given by

$$\alpha\equiv\frac{1}{2\pi}F_{12}(x)=\frac{Q}{L^2}. \tag{3}$$

Hence Q is nothing but the topological charge [1] of the lattice Abelian gauge field. The parameter t continuously

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[1]The axial anomaly on finite lattices is also investigated in Ref. [7] using powerful cohomological arguments [8].

[2]That this kind of reduction of the dimensionality occurs also in higher dimensions is noted in Ref. [9].

connects the link variables with various uniform magnetic fields belonging to different topological sectors classified by the integer topological charge [1,5,6].

The characteristics of the spectrum of HWDO found in [6] can be summarized as follows: (1) The eigenvalues at integral values of t are separated by several gaps and form clusters. (2) For noninteger t the H has in general $2L^2$ distinct eigenvalues and rearrangements of eigenvalues between the clusters occur in a characteristic way as t increases continuously from an integer to the next integer. By carefully inspecting the spectral flows one realizes that for special integer values of $t=r$ with r being an arbitrary divisor of L each eigenvalue is exactly r-ply degenerate. This can be proven by noting that H can be block diagonalized into r $2sL\times 2sL$ matrices, each describing a one-dimensional lattice system of degrees of freedom $2sL$ [6]. In what follows we will generalize the argument of Ref. [6] concerning the characterization of the spectrum of H for special values $t=r$ or $t=L^2/r$ to arbitrary integer values of t. In doing this it is helpful to note the following general facts: (1) The eigenvalues λ of H are bounded by $|\lambda|\leqslant|2-m|+2$ [12]. (2) The spectrum of H is periodic in t with a period L^2. (3) The spectrum is an odd function in t. Hence, it suffices to analyze the eigenvalue spectrum for $0\leqslant t\leqslant L^2/2$.

We first analyze the spectrum for t being an integer multiple of L. Let L be a product of two positive integers r and s. Then t can be expressed as $t=nL^2/r=nsL$, where n is an integer relatively prime with r. Then we have $U_1(x)=1$ and $U_2(x)=\exp[2\pi i n x_1/r]$. Since the link variables are independent of x_2 and are periodic in x_1 with a period r, we can simplify the eigenvalue problem for H by the following Fourier transformation:

$$\varphi(y;p,q)=\frac{1}{sL}\sum_{l=0}^{s-1}\sum_{x_2=0}^{L-1}e^{-iql-ipx_2}\psi(rl+y,x_2), \tag{4}$$

where y ranges from 0 to $r-1$. The Fourier momenta p and q are given by

$$p=\frac{2\pi}{L}k, \qquad q=\frac{2\pi}{s}j$$
$$(k=0,1,\ldots,L-1, \quad j=0,1,\ldots,s-1. \tag{5}$$

H is then block diagonalized into sL $2r\times 2r$ Hermitian matrices $h(p,q)$ given by

$$h(p,q)=\begin{pmatrix} B(p,q) & C(p,q)\\ C^\dagger(p,q) & -B(p,q)\end{pmatrix}, \tag{6}$$

where the first (second) row acts on the upper (lower) component of $\varphi(y;p,q)$. $B(p,q)$ and $C(p,q)$ are defined by

$$[B(p,q)]_{y,y'}=-\frac{1}{2}\delta^{(q)}_{y+1,y'}+\left\{1-\cos\left(p+\frac{2\pi ny}{r}\right)\right\}\delta^{(q)}_{y,y'} -\frac{1}{2}\delta^{(q)}_{y,y'+1},$$

$$[C(p,q)]_{y,y'}=\frac{1}{2}\delta^{(q)}_{y+1,y'}+\sin\left(p+\frac{2\pi ny}{r}\right)\delta^{(q)}_{y,y'} -\frac{1}{2}\delta^{(q)}_{y,y'+1}. \tag{7}$$

The $\delta^{(q)}_{y,y'}$ is the Kronecker δ symbol for $0\leqslant y,y'\leqslant r-1$ and satisfies the twisted boundary conditions

$$\delta^{(q)}_{y,r}=e^{-iq}\delta_{y,0}, \qquad \delta^{(q)}_{r,y}=e^{iq}\delta_{y,0}. \tag{8}$$

We thus find that the original two-dimensional system of degrees of freedom $2L^2$ is decomposed into sL one-dimensional systems of degrees of freedom $2r$, each described by $h(p,q)$. The corresponding eigenvalue problem is just a relativistic analogue of the Harper equation [10].

The eigenvalues of H are determined by the secular equation $\det[h(p,q)-\lambda]=0$. It takes the form

$$\det[h(p,q)-\lambda]=f_r^{(n)}(\lambda;p)-\frac{(-1)^{r-1}}{2^{r-4}}\sin^2\frac{rp}{2}\sin^2\frac{q}{2}=0, \tag{9}$$

where $f_r^{(n)}(\lambda;p)=\lambda^{2r}+\cdots$ is a polynomial of order $2r$ and is defined by

$$f_r^{(n)}(\lambda;p)=\det[h(p,0)-\lambda]. \tag{10}$$

The q-dependent term in Eq. (9) can be easily found from the explicit form of $h(p,q)$. For n and r being relatively prime integers $h(p,0)$ and $h(0,0)$ are related by an orthogonal transformation and $f_r^{(n)}(\lambda;p)$ then becomes independent of p. We simply write it as $f_r^{(n)}(\lambda)$. Explicit forms of $f_r^{(1)}(\lambda)$ are given in Ref. [6] for $r=1,\ldots,6$.

Later we need to consider Eq. (10) for n and r not necessarily being relatively prime integers. In general $f_r^{(n)}(\lambda;p)$ depends on p and satisfies the factorization property

$$f_r^{(n)}(\lambda;p)=\prod_{j=0}^{s'-1}\left(f_{r'}^{(n')}(\lambda)-\frac{(-1)^{r'-1}}{2^{r'-4}}\sin^2\frac{r'p}{2}\sin^2\frac{\pi j}{s'}\right), \tag{11}$$

where $n'=n/s'$ and $r'=r/s'$ with s' being the greatest common divisor of n and r. The easiest way to show this is to consider the case of $r=L$ and $t=nL^2/r=n'L^2/r'=n's'L$. Since $s=1$, the only allowed value of q is 0. The secular equation (9) then becomes $f_r^{(n)}(\lambda;p)=0$. On the other hand, the same set of eigenvalues must be reproduced by the expressions (9) with the substitutions $r,n\to r',n'$ and $q\to 2\pi j/s'$ for $j=0,\ldots,s'-1$. The identity (11) then follows from these.

We have shown that the eigenvalues of H for t being any integer multiple of L are completely characterized by a set of functions $f_r^{(n)}(\lambda)$. We now extend the results to an arbitrary integer t. Let n and s be relatively prime positive integers such that $t/L=n/s$, then we may find a positive integer r satisfying $t=nr$ and $L=rs$. Denoting x_2 by two nonnegative integers y and l ($0\leqslant y<s$, $0\leqslant l<r$) as $x_2=sl+y$,

we see that the link variables (2) are independent of l or, equivalently, periodic in y with a period s. We then introduce new variables $\Psi(z;q)$ by

$$\Psi(z;q)=\frac{1}{L}\sum_{y=0}^{s-1}\sum_{l=0}^{r-1} e^{-i(py+ql)}\psi(x_1,sl+y), \tag{12}$$

where q, p and z are defined by

$$q=\frac{2\pi j}{r},\quad p=\frac{2\pi nk}{s}+\frac{2\pi j}{L},\quad z=kL+\overline{x_1}$$
$$(j=0,\ldots,r-1,\ k=0,\ldots,s-1). \tag{13}$$

Since the shift $z\rightarrow z+sL$ simply corresponds to $k\rightarrow k+s$ or, equivalently, to $p\rightarrow p+2\pi n$, we may regard $\Psi(z;q)$ as functions of z with a period sL. Here a miracle occurs. In the new variables $\Psi(z;q)$ defined on the one-dimensional periodic lattice, H is again block diagonalized into r Hermitian $2sL\times 2sL$ matrices of the form (6) and (7). The concrete expressions of B and C are obtained from Eq. (7) by making the following substitutions:

$$y,\ r,\ p,\ q \quad\rightarrow\quad z,\ sL,\ \frac{q}{s},\ 0. \tag{14}$$

The secular equations for the eigenvalues of H at $t=nr$ are then given by $f_{sL}^{(n)}(\lambda;q/s)=0$. Noting the factorization formula (11), we find that all the eigenvalues of H at $t=nr$ are determined by the following set of secular equations:

$$f_{r's^2}^{(n')}(\lambda)=\frac{(-1)^{r's^2-1}}{2^{r's^2-4}}\sin^2\frac{\pi js}{s'}\sin^2\frac{\pi l}{s'}$$
$$(j=0,1,\cdots,r-1,\ l=0,\cdots,s'-1), \tag{15}$$

where $n'=n/s'$ and $r'=r/s'$ with s' being the greatest common divisor of r and n.

We see from Eq. (15) that for each j and l there are $2r's^2$ distinct eigenvalues and they must lie one by one in the $2r's^2$ narrow intervals given by the inequality

$$0\leqslant(-1)^{r's^2-1}2^{r's^2-4}f_{r's^2}^{(n')}(\lambda)\leqslant 1. \tag{16}$$

Each interval contains exactly rs' eigenvalues. Furthermore, the intervals themselves form roughly $2r's^2/n'$ clusters. This explains the characteristic features of the spectral flows of H found by the numerical investigation [6]. We can also find the multiplicity of the eigenvalues from Eq. (15). In particular, the multiplicity at $t=rn$ with n and r being relatively prime is exactly r. In Fig. 1 the spectrum of H for the uniform magnetic field is shown. One can easily see that the feather-shape gaps form a fractal pattern. It looks quite different from the butterfly diagram [11]. This is due to the special choice of $m=1$ and the butterflylike gaps appear for $m\neq 1$. The spectrum for $m=1/2$ is shown in Fig. 2 for comparison.

In order to understand the appearance of the fractal structure of the spectrum we consider the infinite volume limit.

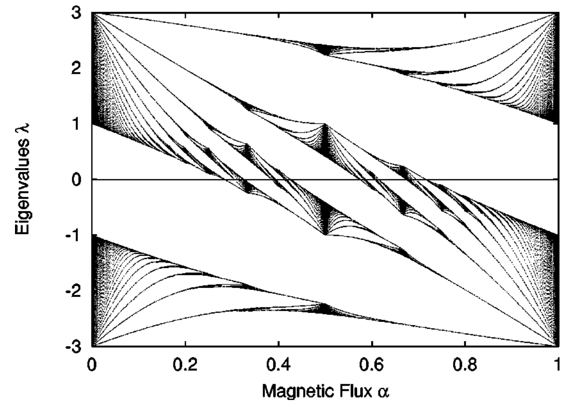

FIG. 1. The spectrum of H for uniform magnetic fields ($L=23$, $m=1$).

The magnetic flux α defined by Eq. (3) may be an arbitrary real number by considering the limit $t,L\rightarrow\infty$ with $\alpha=t/L^2$ fixed. In particular the eigenvalues at $\alpha=n/r$ with r and n being mutually prime positive integers form $2r$ bands given by the inequality

$$0\leqslant(-1)^{r-1}2^{r-4}f_r^{(n)}(\lambda)\leqslant 1, \tag{17}$$

where n specifies how these bands cluster each other. Roughly speaking, if $2n<r$ or $r-n$ if $2n>r$ then n stands for the number of neighboring bands. An irrational flux is realized as an appropriate limit $r,n\rightarrow\infty$. This implies that the finite number of bands for a rational flux split into an infinite number of tiny bands (maybe a Cantor set). Though the spectrum appears smoothly varying with the flux due to the low resolution of the plot, such tremendous splittings and focusings of the bands take place continually during a small change of the flux.

We now turn to the computation of the index of the ODO defined by

$$D=1+\sigma_3\frac{H}{\sqrt{H^2}}. \tag{18}$$

It is given by [3]

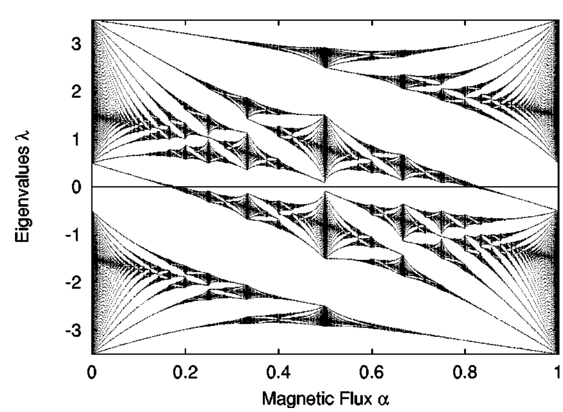

FIG. 2. The spectrum of H for uniform magnetic fields ($L=23$, $m=1/2$).

$$\mathrm{index}D = \mathrm{Tr}\,\sigma_3\left(1-\frac{1}{2}D\right) = -\frac{1}{2}\mathrm{Tr}\frac{H}{\sqrt{H^2}}, \tag{19}$$

where Tr implies the sum over the lattice coordinates as well as the trace over the spin indices. Since the trace on the right-hand side (RHS) of this expression equals the number of positive eigenvalues of H minus the number of negative eigenvalues, we can find index$D$ for an arbitrary uniform magnetic field by counting the root asymmetry of the secular equation (15), where by root asymmetry of a polynomial equation we mean the number of positive roots minus the number of negative roots. In general the origin $\lambda=0$ lies outside of the intervals defined by the inequalities (16) as can be seen from Fig. 1. If this is satisfied, it is possible to relate index$D$ with the root asymmetry of $f^{(n')}_{r's^2}(\lambda)=0$. We thus obtain for $t=nr=n'r's'^2$

$$\mathrm{index}D = -\frac{1}{2}rs'\sigma^{(n')}_{r's^2}, \tag{20}$$

where $\sigma^{(n)}_r$ stands for the root asymmetry of $f^{(n)}_r(\lambda)=0$.

To understand $\sigma^{(n)}_r$ the roots of $f^{(1)}_r(\lambda)=0$ are plotted in Fig. 3. As is easily seen, the root with the minimum absolute value for each r monotonically increases as r and changes the sign from minus to plus at $r=4$. We thus find $\sigma^{(1)}_r=0$ for $r\leqslant 3$ and $\sigma^{(1)}_r=2$ for $r\geqslant 4$. In the case $n>1$ the behaviors of $\sigma^{(n)}_r$ are rather complicated for $r<4n$. However, we know $\sigma^{(n)}_r=0$ for $r=2n$, $3n$ and $\sigma^{(n)}_r=2n$ for $r=4n$, $5n$, ... from the factorization relation (11). In fact it is possible to show $\sigma^{(n)}_r=2n$ for sufficiently large r ($\geqslant 4n$) by directly evaluating the spectral asymmetry of the Hermitian matrix (6) for $p=q=0$ in the large r limit [13]. We thus obtain for $r's^2\geqslant 4n'$ or equivalently for $\alpha=t/L^2\leqslant 1/4$

$$\mathrm{index}D = -rs'n' = -rn = -\frac{1}{2\pi}\sum_x F_{12}(x), \tag{21}$$

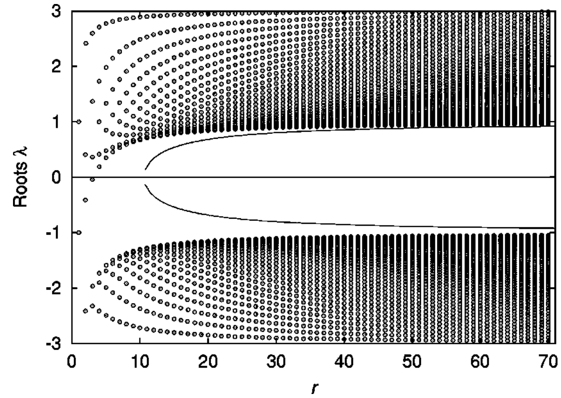

FIG. 3. Roots of $f^{(1)}_r(\lambda)=0$ for $1\leqslant r\leqslant 70$. The two curves indicate the known bound $|\lambda|\geqslant\sqrt{1-(2+\sqrt{2})\sin(\pi/r)}$.

where use has been made of Eq. (3). This is the index theorem for the ODO (18) in the Abelian gauge background in two dimensions [5,6].

We have shown that the system described by the HWDO for an arbitrary uniform magnetic field in two dimensions can be cast into one-dimensional systems described by the relativistic Harper equations. From the spectral asymmetry of the relativistic Harper system we have succeeded in determining the index of the ODO and established the index theorem for the uniform magnetic field $|F_{12}(x)|\leqslant\pi/2$. The bound is of course not optimal and depends on the choice of the parameter m. The restriction to uniform magnetic fields is not essential for the index theorem. The invariance of the index and the topological charge under an arbitrary infinitesimal variation of the link variables implies that the index theorem (20) holds also for lattice gauge fields sufficiently close to the configurations, giving rise to the uniform magnetic field. The coincidence between the index and the topological charge, however, is in general violated for $|F_{12}(x)|>\pi/2$. Our results are of help in distinguishing smooth gauge field configurations from nonsmooth ones on finite lattices.

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